# Engineering Notes

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## Solutions of True Proportional Navigation for Maneuvering and Nonmaneuvering Targets

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### I. Introduction

IN true proportional navigation (TPN), the commanded acceleration is applied in a direction normal to the missile-target line of sight (LOS) and its magnitude is proportional to the product of the angular rate of the LOS times the closing speed between missile and target. In earlier studies, only nonmaneuvering targets were considered[1] and closing speed was assumed to be a constant to simplify the formulation.[2] Thus, a simple closed-form solution was able to be obtained as a function of time-to-go. Then, the variation of closing speed was considered during formulation, but it is still not used in the generation of commanded acceleration from TPN.[3-8] Therefore, only a complicated and incomprehensible closed-form solution without further discussion was obtained.[3]

In this Note, the TPN in which the commanded acceleration is proportional to the product of LOS rate times variable closing speed is studied. The closed-form solutions will be derived for both maneuvering and nonmaneuvering targets. Some important characteristics, related to the system performance and the capture capability, are introduced and discussed in detail. The closed-form solution can be simply expressed, in general, as a function of the range-to-go. For the special case of a nonmaneuvering target, the solution as a function of LOS angle can be obtained through transformation.

### II. Closed-Form Solution for Nonmaneuvering Target

Consider a missile of speed $V_M$ is pursuing a nonmaneuvering target wtih speed $V_T$ under the guidance law of TPN, as shown in Fig. 1. Here, the commanded acceleration is given in the direction normal to LOS and its magnitude is

$$a_c = \lambda \dot{r} \dot{\theta} \tag{1}$$

where $\lambda$ is the effective proportional navigation constant, $\dot{r}$ the rate of range-to-go between missile and target, and $\dot{\theta}$ the angular rate of LOS. The equations of relative motion between missile and target are

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{2a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_c = \lambda \dot{r} \dot{\theta} \tag{2b}$$

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If $r_0$, $\dot{r}_0$, and $\dot{\theta}_0$ are the initial conditions of $r$, $\dot{r}$, and $\dot{\theta}$, respectively, it can be derived that

$$\dot{\theta} = \dot{\theta}_0 \left( \frac{r}{r_0} \right)^{\lambda - 2} \tag{3a}$$

$$\dot{r} = \dot{r}_0 \sqrt{1 - B^2[1 - (r/r_0)^{2\lambda - 2}]}, \qquad B = \frac{r_0 \dot{\theta}_0}{\sqrt{\lambda - 1}\, \dot{r}_0} \tag{3b}$$

For the purpose of effective intercept, $B^2$ must be $<1$, i.e.,

$$(\lambda - 1)A^2 = \frac{1}{B^2} > 1, \qquad A = \frac{\dot{r}_0}{r_0 \dot{\theta}_0} \tag{4}$$

and the condition $\lambda > 1$ is also included implicitly. The commanded acceleration $a_c$ for the missile can be written as

$$a_c = \lambda \dot{r} \dot{\theta} = \lambda \dot{r}_0 \dot{\theta}_0 \left( \frac{r}{r_0} \right)^{\lambda - 2} \sqrt{1 - B^2[1 - (r/r_0)^{2\lambda - 2}]} \tag{5}$$

which will approach infinity when $\lambda < 2$ and will approach zero when $\lambda > 2$ during intercept period. The capture time, denoted by $T$, is

$$\frac{T}{-(r_0/\dot{r}_0)} = \int_0^1 \frac{\mathrm{d}x}{\sqrt{B^2 x^{2\lambda - 2} + (1 - B^2)}} \tag{6}$$

Finally, the relation between $\theta$ and $r$ can be expressed as

$$\theta = \frac{1}{\sqrt{\lambda - 1}} \ln \frac{\sqrt{B^2 (r/r_0)^{2\lambda - 2} + (1 - B^2)} + B(r/r_0)^{\lambda - 1}}{1 + B} \tag{7}$$

Furthermore, the inverse function of Eq. (7) can be written as

$$\frac{r}{r_0} = \left[ \cosh\left(\sqrt{\lambda - 1}\,\theta\right) + \frac{1}{B} \sinh\left(\sqrt{\lambda - 1}\,\theta\right) \right]^{1/(\lambda - 1)} \tag{8}$$

Then

$$\dot{\theta} = \dot{\theta}_0 \left[ \cosh\left(\sqrt{\lambda - 1}\,\theta\right) + \frac{1}{B} \sinh\left(\sqrt{\lambda - 1}\,\theta\right) \right]^{(\lambda - 2)/(\lambda - 1)} \tag{9a}$$

$$r\dot{\theta} = r_0 \dot{\theta}_0 \left[ \cosh\left(\sqrt{\lambda - 1}\,\theta\right) + \frac{1}{B} \sinh\left(\sqrt{\lambda - 1}\,\theta\right) \right] \tag{9b}$$

$$\dot{r} = \dot{r}_0 \left[ \cosh\left(\sqrt{\lambda - 1}\,\theta\right) + B \sinh\left(\sqrt{\lambda - 1}\,\theta\right) \right] \tag{9c}$$

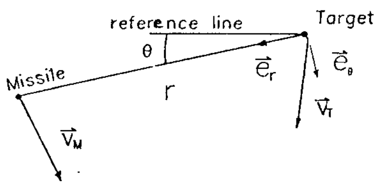


**Fig. 1 Planar pursuit geometry.**

## III. Closed-Form Solution for Maneuvering Target

For the case of a maneuvering target, we assume that the target maneuver is always in the direction normal to the LOS. We then have the equations of relative motion between missile and target:

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{10a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \lambda\dot{r}\dot{\theta} - a_T \tag{10b}$$

where $a_T$ is the magnitude of the target acceleration and is regulated proportional to the closing speed as $a_T = -b\dot{r}$ ($b$ is a constant) to simplify the formulation. From Eq. (10b), it is seen that $a_T$ can reduce the effectiveness of the commanded missile acceleration. Multiplying Eq. (10b) by $r/\dot{r}$ and rearranging it gives

$$\frac{\mathrm{d}h}{\mathrm{d}r} = \lambda\frac{h}{r} - a_T\frac{r}{\dot{r}} = \lambda\frac{h}{r} + br \tag{11}$$

where $h(=r^2\dot{\theta})$ is the massless angular momentum of missile relative to the target point. The solution of Eq. (11) can be obtained as

$\lambda \neq 2$:

$$h = c_1 r^\lambda + \frac{b}{2-\lambda}r^2 \tag{12a}$$

$$\dot{\theta} = c_1 r^{\lambda-2} + \frac{b}{2-\lambda} \tag{12b}$$

$\lambda = 2$:

$$h = c_2 r^2 + br^2 \ln r \tag{12c}$$

$$\dot{\theta} = c_2 + b\ln r \tag{12d}$$

where $c_1$ and $c_2$ are constants of integration. We find that, with a maneuvering target, $\dot{\theta}$ will never approach zero as $r$ approaches zero for any value of $\lambda$. From the initial condition, Eqs. (12b) and (12d) can be written as

$\lambda \neq 2$:

$$\dot{\theta} = \dot{\theta}_0\left\{\left(\frac{r}{r_0}\right)^{\lambda-2} + \frac{c}{\lambda-2}\left[1 - \left(\frac{r}{r_0}\right)^{\lambda-2}\right]\right\} \tag{13a}$$

$\lambda = 2$:

$$\dot{\theta} = \dot{\theta}_0\left[1 - c\ln\left(\frac{r}{r_0}\right)\right] \tag{13b}$$

where

$$c = -\frac{b}{\dot{\theta}_0} = \frac{a_{T_0}}{\dot{r}_0\dot{\theta}_0}$$

Thus, the response of LOS angular rate can be separated into two parts: one is induced by the initial angular rate of LOS and the other is induced by the maneuver of target. For $\lambda \leq 2$, $\dot{\theta}$ will approach infinity during intercept period and will approach $b/(2-\lambda)$ with $\lambda > 2$. Figure 2 shows a typical variation between $\dot{\theta}$ and $r$ for different values of $\lambda$ with maneuvering target. Substituting Eqs. (13) into Eq. (10a) we have

$\lambda \neq 2$:

$$\ddot{r} = r\dot{\theta}^2 = r_0\dot{\theta}_0^2\left(\frac{r}{r_0}\right)\left\{\left[1 - \frac{c}{\lambda-2}\right]\left(\frac{r}{r_0}\right)^{\lambda-2} + \frac{c}{\lambda-2}\right\}^2 \tag{14a}$$

$\lambda = 2$:

$$\ddot{r} = r\dot{\theta}^2 = r_0\dot{\theta}_0^2\left(\frac{r}{r_0}\right)\left[1 - c\ln\left(\frac{r}{r_0}\right)\right]^2 \tag{14b}$$

Then the following differential equations can be obtained:

$\lambda \neq 2$:

$$\begin{aligned}\mathrm{d}\left(\frac{\dot{r}^2}{2}\right) &= \ddot{r}\,\mathrm{d}r \\ &= r_0\dot{\theta}_0^2\left\{\left[1 - \frac{c}{\lambda-2}\right]^2\left(\frac{r}{r_0}\right)^{2\lambda-3} + \frac{2c}{\lambda-2}\right. \\ &\quad \left.\times\left[1 - \frac{c}{\lambda-2}\right]\left(\frac{r}{r_0}\right)^{\lambda-1} + \frac{c^2}{(\lambda-2)^2}\left(\frac{r}{r_0}\right)\right\}\mathrm{d}r\end{aligned} \tag{15a}$$

$\lambda = 2$:

$$\begin{aligned}\mathrm{d}\left(\frac{\dot{r}^2}{2}\right) &= \ddot{r}\,\mathrm{d}r \\ &= r_0\dot{\theta}_0^2\left\{\left(\frac{r}{r_0}\right) - 2c\left(\frac{r}{r_0}\right)\ln\left(\frac{r}{r_0}\right)\right. \\ &\quad \left.+ c^2\left(\frac{r}{r_0}\right)\left[\ln\left(\frac{r}{r_0}\right)\right]^2\right\}\mathrm{d}r\end{aligned} \tag{15b}$$

Because $\lambda \leq 1$ is meaningless, as described earlier, now only $\lambda > 1$ is considered. The solutions of Eqs. (15) are

$\lambda \neq 2$:

$$\begin{aligned}\dot{r}^2 &= r_0^2\dot{\theta}_0^2\left\{\frac{1}{\lambda-1}\left[1 - \frac{c}{\lambda-2}\right]^2\left(\frac{r}{r_0}\right)^{2\lambda-2}\right. \\ &\quad \left.+ \frac{4c}{\lambda(\lambda-2)}\left[1 - \frac{c}{\lambda-2}\right]\left(\frac{r}{r_0}\right)^{\lambda} + \frac{c^2}{(\lambda-2)^2}\left(\frac{r}{r_0}\right)^2\right\} \\ &\quad + \dot{r}_0^2 - \frac{r_0^2\dot{\theta}_0^2}{\lambda-1}\left\{1 + \frac{1}{\lambda}[(1+c)^2 - 1]\right\}\end{aligned} \tag{16a}$$

$\lambda = 2$:

$$\begin{aligned}\dot{r}^2 &= r_0^2\dot{\theta}_0^2\left\{\left(1 + c + \frac{1}{2}c^2\right)\left(\frac{r}{r_0}\right)^2 - (2c + c^2)\left(\frac{r}{r_0}\right)^2\ln\left(\frac{r}{r_0}\right)\right. \\ &\quad \left.+ c^2\left(\frac{r}{r_0}\right)^2\left[\ln\left(\frac{r}{r_0}\right)\right]^2\right\} + \dot{r}_0^2 - r_0^2\dot{\theta}_0^2\left[\frac{1}{2} + \frac{1}{2}(1+c)^2\right]\end{aligned} \tag{16b}$$

From the results in Eq. (16), we find that the maneuver of target also affects the variation of the closing speed, as shown

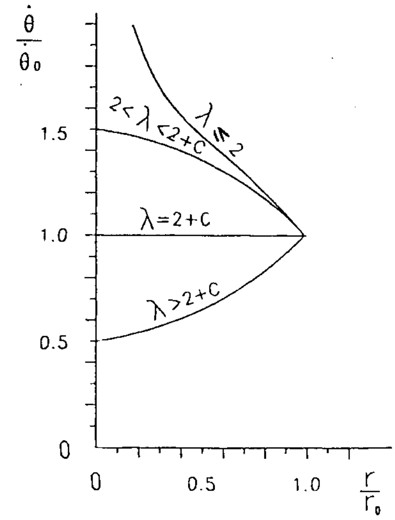


**Fig. 2** $\dot{\theta}$ **vs** $r$ **for different values of** $\lambda$ **with maneuvering target.**

in Fig. 3. Also, the final closing speed at intercept can be written as

$$\dot{r}_f = \dot{r}_0\sqrt{1 - B^2\left\{1 + \frac{1}{\lambda}[(1+c)^2 - 1]\right\}} \tag{17}$$

Thus, the following constraint must be satisfied for effective intercept of target:

$$(\lambda - 1)A^2 > 1 + \frac{1}{\lambda}[(1+c)^2 - 1] \tag{18}$$

It will be the same form as shown in Eq. (4) if $c = 0$. Figure 4 shows the boundary of capture area under the effect of target maneuver. In this case, the commanded acceleration at final intercept point will not approach zero for any value of $\lambda$. When $\lambda \leq 2$, it goes to infinity, obviously. When $\lambda > 2$, it approaches a constant value related to the target maneuver, i.e.,

$$a_{c_f} = \lambda \dot{r}_f \dot{\theta}_f = \frac{\lambda}{\lambda - 2} a_T \tag{19}$$

The increase in the value of $\lambda$ can decrease the required maneuverability of the missile for pursuing a maneuvering target. For example, $a_{c_f}$ is three times of $a_T$ with $\lambda = 3$ and is two times of $a_T$ with $\lambda = 4$.

## IV. Discussion

For the case of a nonmaneuvering target, all of the solutions and system characteristics can be obtained from the case of a maneuvering target with $c = 0$. From the previous analysis, $\lambda$ must be chosen to be $>2$ to avoid $a_c$ going to infinity, and $A^2$ is usually constrained to be $\geq 1$ to simplify the capture criterion to $\lambda > 2$. Then the magnitude of missile initial velocity, with its direction heading to the target point, must be $\geq \sqrt{2}$ times of target speed, in general, to guarantee that $A^2 \geq 1$ for all aspect capability.

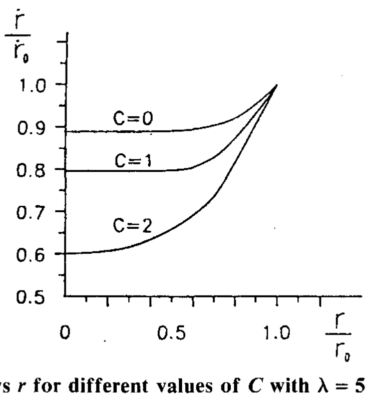


Fig. 3 $\dot{r}$ vs $r$ for different values of $C$ with $\lambda = 5$ and $A = 1$.

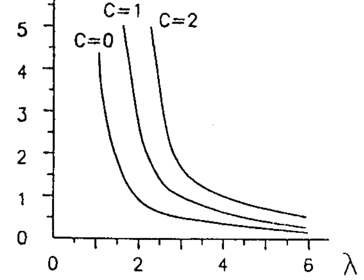


Fig. 4 Capture area for different values of $C$.

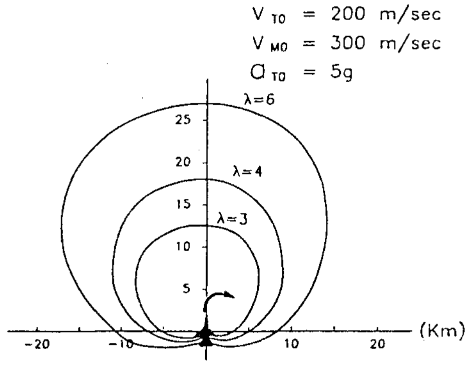


Fig. 5 Typical firing envelopes with different values of $\lambda$ for a maneuvering target.

For the case of a maneuvering target, the target maneuver also affects the system performance. From Eq. (18), we have

$$\lambda > \frac{(A^2+1) + \sqrt{(A^2+1)^2 + 4A^2[(1+c)^2 - 1]}}{2A^2} \tag{20}$$

If $A^2 \geq 1$ exists in actual initial intercept conditions, then the capture criterion can be simplified to

$$\lambda > 2 + c \tag{21}$$

with $A^2 = 1$, and, thus, a larger value of $\lambda$ is required for larger maneuver of the target. In this case, both the responded LOS rate and the commanded acceleration are monotonously decreased until intercept, but will not approach zero due to the effect of target maneuver. Also, from Eq. (18), a firing envelope exists for a maneuvering target, which can be derived with the relation $c = (a_{T_0} r_0)/(v_{r_0} v_{\theta_0})$,

When $\psi < 180$ deg:

$$r_0 < \frac{v_{r_0} v_{\theta_0}}{a_{T_0}}\left[\sqrt{(\lambda - 1)(\lambda A^2 - 1)} - 1\right] \tag{22a}$$

When $\psi > 180$ deg:

$$r_0 < -\frac{v_{r_0} v_{\theta_0}}{a_{T_0}}\left[\sqrt{(\lambda - 1)(\lambda A^2 - 1)} + 1\right] \tag{22b}$$

where $\psi$ is the aspect angle and is defined to be zero in tail chase condition. A typical example of firing envelope for a maneuvering target with different values of $\lambda$ is shown in Fig. 5.

## V. Conclusions

In this study, the exact closed-form solutions of the differential equations describing the relative motion of a missile pursuing maneuvering and nonmaneuvering targets are derived, according to true proportional navigation guidance law. For the case of a nonmaneuvering target, it is found that the effective proportional navigation constant should be $>2$ with the ratio of relative initial radial speed to tangential speed $\geq 1$. Then, the missile-to-target speed ratio at initial instant should be $\geq \sqrt{2}$ for all aspect capabilities. For the case of a maneuvering target, a larger value of effective proportional navigation constant is required for effective intercept of target. Simultaneously, a firing envelope exists in this case due to target maneuver.

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# Ground-Based Implementation and Verification of Control Laws for Tethered Satellites

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## Introduction

THE getaway tether experiment (GATE) is a single tether satellite system that will test and develop control technology for tethered system. Originally a free-flying tethered system released from a getaway special canister on the orbiter,[1] the system is now one experiment in the tether dynamics explorer (TDE) series launched by a Delta II.[2] The system consists of a Delta II second stage and subsatellite connected by a single tether. The subsatellite contains a motorized reel mechanism that will be the primary means of control actuation.[3]

A model of a single tether system that considered only length and in-plane libration dynamics was developed by Rupp[4] along with a tether tension control law requiring tether length/length rate feedback. Baker et al.[5] expanded this tension control law and performed simulations using a system model that included out-of-plane libration dynamics. Greene et al.[1] have applied a tension control law of the form proposed by Rupp to the GATE system. Simulation results demonstrated that the tension control law produced desirable results in the stabilization of the GATE system during deployment and retrieval maneuvers.

Modi and Misra[6] have developed extensive models of tethered systems and have performed simulations using control laws consisting of different forms of length-rate control. Recently, Davis and Banerjee[7] have expanded an idea introduced by Baker et al. in which out-of-plane librations are damped using a length-rate control after the tether has been deployed, or retrieved, to the desired length. A yo-yo type motion was used to perform out-of-plane libration damping requiring very high deployment and retrieval rates in relation to the frequency of the out-of-plane libration.

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The present work investigates the implementation of these control schemes to prototype hardware designed for space flight. A prototype reel mechanism has been constructed for use in the GATE system and ground tested. Both the out-of-plane libration control scheme proposed by Davis and Banerjee and a converted tension control law have been implemented with tether length and length rate available as feedback.

## Prototype Design

The reel mechanism consists of a spool and a level wind mechanism that are driven by independent stepping motors. The spool is designed to hold 1.8 km of 0.075-cm-diam tether. The entire structure of dimensions 41.3 × 30.5 × 21.0 cm (16.25 × 12.0 × 8.25 in.) fits in the TDE experimental envelope. The mechanism is shown in Fig. 1.

Figure 2 is a simplified block diagram of the control system. Two stepping motors are used to eliminate any mechanical connections between the level wind mechanism and the spool shaft. Each motor driver requires a pulse frequency-modulated (PFM) signal to command the desired stepping rate. The stepper motor coupled to the level wind is electronically geared down by dividing the PFM signal frequency by three. Optical switches are used to trigger logic circuits providing the level wind motor with direction signals based on the spool motor direction and present level wind motor direction. An optical encoder connected to the spool shaft provides length and length-rate feedback. Pulses from the encoder produce a count that can be related to instantaneous tether length. The encoder signals also provide a direction of rotation signal, via logic circuitry, which is used as an up/down signal to the counter circuitry.

An IBM PC is used to control the reel mechanism. Interrupt-based control software was written in C language. At each interrupt, the tether length and length rate are calculated and new commands for the motors are formulated before interrupts are re-enabled. The motor commands consist of a motor step rate and direction of shaft rotation. The motor step-rate commands are transformed into a count word that is used by an 8253 timer to produce the PFM signal required by the motor driver.

## System Equations

For testing purposes, the prototype was configured such that the tether is deployed toward the Earth. A mass of 22 g was placed on the end of the tether to produce a static tension in the tether, which is slightly larger than the static tension the actual system will experience when fully deployed.[2] In such a configuration, the equations of motion governing the pendu-

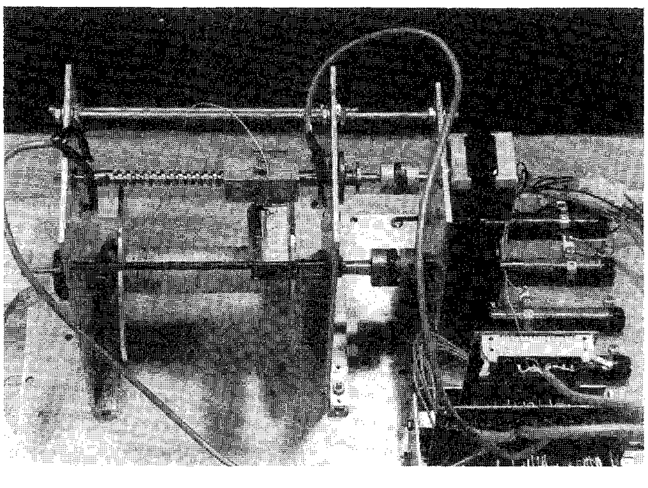

**Fig. 1 Photograph of the reel/deployer mechanism. Note the two stepper motors: one is used to drive the reel while the other drives a spooling mechanism for level wind.**